

Spherical Gravitating Systems of Arbitrary Dimension

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We study spherically symmetric solutions to the Einstein field equations under the assumption that the space-time may possess an arbitrary number of spatial dimensions. The general solution of Synge is extended to describe systems of any dimension. Arbitrary dimension analogues of four dimensional solutions are also presented, derived using the above scheme. Finally, we discuss the requirements for the existence of Birkhoff's theorems in space-times of arbitrary dimension with or without matter fields present. Cases are discussed where the assumptions of the theorem are considerably weakened yet the theorem still holds. We also discuss where the weakening of certain conditions may cause the theorem to fail.

§1. Introduction

Space-times possessing dimension greater than four have been of much interest at least since the pioneering ideas of Kaluza¹⁾ and Klein^{2) 3)}. Since then there have been numerous theories of unification (for example superstring theory), many of which require more than three spatial dimensions to be consistent. The low energy sector of many of these theories reduce to a multi-dimensional General Relativity theory as is studied here.

Although extra dimensions are usually thought to be compact, the extra dimensions may be manifest on scales which are relevant when studying cosmological systems near the big bang or gravitational collapse approaching the singularity. Also, there has lately been much interest in the possibility of large extra dimensions^{4)–6)} in which higher dimensional effects may be observed at relatively low TeV scales. Microscopic systems above these scales may be formed from the collapse of large, effectively four dimensional, initial conditions. If these scenarios do indeed describe our universe, then gravitating systems at these scales will behave as higher dimensional systems and deviate from the predictions of four dimensional physics.

Much interesting work has been done in the field of higher dimensional gravity (for example, see^{7)–14)} and references therein. The excellent review by Melnikov¹⁰⁾ has an extensive list of references). Usually, these studies involve specific matter fields or a class of metrics such as the FRW cosmological metrics. However, little work has been done on a reasonably general methodology which may be utilised in studying higher dimensional gravitation problems. Granted, a completely general method to solve Einstein's field equations does not exist and therefore in this note

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we focus on spherically symmetric systems. Since some interesting studies have been performed on the subject of higher dimensional black holes (for examples see ^{16), 17)} and references therein) our concentration here will be on non-vacuum systems.

Finally, studies of General Relativity in arbitrary dimension will also serve to shed light on the theory's internal consistencies and lead to a greater understanding of the theory as a whole. This avenue has already proved fruitful in the case of low dimensional black holes ^{18)–22)}, for example. ^{*)}

We therefore believe that it is of much interest to study how the presence of an arbitrary number of spatial dimensions affects the solutions of general relativity. To this end, in section 2, we generalise Synge's ²³⁾ method of solving the spherically symmetric Einstein field equations so that it may apply to D -dimensional systems. In section 3 we present solutions which are arbitrary dimensional counterparts to some important 4 dimensional solutions. Finally, in section 4, we rigorously prove a D -dimensional staticity or Birkhoff's theorem and comment on situations when the theorem can fail.

§2. The general solution

The fundamental equations governing the space-time geometry may be derived from the action ^{**)}

$$S = \int \left(-\frac{R}{16\pi} + \mathcal{L}_m \right) \sqrt{g} d^D x, \quad (2.1)$$

where \mathcal{L}_m is the matter Lagrangian density and the positive integer D is assumed to be larger than 2.

The action principle with suitable boundary conditions gives rise to the D -dimensional Einstein field equations:

$$R^\mu_\nu - \frac{1}{2} R \delta^\mu_\nu = 8\pi T^\mu_\nu, \quad (2.2)$$

along with supplementary equations governing the behaviour of the matter fields. We wish to study solutions under the ansatz of spherical symmetry. In curvature coordinates this allows us to write the space-time metric as:

$$ds^2 = -e^{\nu(r,t)} dt^2 + e^{\lambda(r,t)} dr^2 + r^2 d\Omega_{(D-2)}^2, \quad (2.3)$$

where $d\Omega_{(D-2)}^2$ is the line element on a unit $D - 2$ sphere:

$$d\Omega_{(D-2)}^2 = \left[d\theta_{(0)}^2 + \sum_{n=1}^{D-3} d\theta_{(n)}^2 \left(\prod_{m=1}^n \sin^2 \theta_{(m-1)} \right) \right]. \quad (2.4)$$

^{*)} The literature on lower dimensional black holes is extensive. We apologize that we cannot cite all the excellent work in this area.

^{**)} Conventions in this paper follow those of ¹⁵⁾ with G_D , the D -dimensional Newton's constant, and $c \equiv 1$. Here, Greek indices take on values $0 \rightarrow D - 1$ whereas Latin indices take on values $1 \rightarrow D - 1$ (spatial indices).

The corresponding hyper-surface volume is given by:

$$V_{(D-2)\text{-sphere}} = \frac{2\pi^{(D-1)/2}}{\Gamma\left(\frac{D-1}{2}\right)}, \quad (2.5)$$

and the coordinate ranges are:

$$t_1 < t < t_2, \quad r_1 < r < r_2, \quad 0 < \theta_{(0)}, \theta_{(1)}, \dots, \theta_{(D-4)} < \pi, \quad 0 \leq \theta_{(D-3)} < 2\pi. \quad (2.6)$$

The above metric yields the following field equations for (2.2):

$$8\pi T_t^t = -\frac{D-2}{2r^2} \left[(D-3) \left(1 - e^{-\lambda(r,t)} \right) + r e^{-\lambda(r,t)} \lambda(r,t)_{,r} \right], \quad (2.7a)$$

$$8\pi T_r^r = -\frac{D-2}{2r^2} \left[(D-3) \left(1 - e^{-\lambda(r,t)} \right) - r e^{-\lambda(r,t)} \nu(r,t)_{,r} \right], \quad (2.7b)$$

$$8\pi T_r^t = -\frac{D-2}{2r} e^{-\nu(r,t)} \lambda(r,t)_{,t}, \quad (2.7c)$$

$$\begin{aligned} 8\pi T_{\theta(A)}^{\theta(A)} = & \frac{e^{-\nu(r,t)}}{4} \left[\nu(r,t)_{,t} \lambda(r,t)_{,t} - (\lambda(r,t)_{,t})^2 - 2\lambda(r,t)_{,t,t} \right] \\ & + \frac{e^{-\lambda(r,t)}}{4} \left[2\nu(r,t)_{,r,r} + (\nu(r,t)_{,r})^2 + \frac{2(D-3)}{r} (\nu(r,t) - \lambda(r,t))_{,r} \right. \\ & \left. - \nu(r,t)_{,r} \lambda(r,t)_{,r} + \frac{2}{r^2} (D-3)(D-4) \right] - \frac{2(D-3)(D-4)}{r^2}. \end{aligned} \quad (2.7d)$$

Enforcing conservation laws, $T_{\nu;\mu}^\mu \equiv 0$, yields the following non-trivial equations:

$$\begin{aligned} T_{t,t}^t + T_{t,r}^r + \frac{1}{2} \lambda(r,t)_{,t} (T_t^t - T_r^r) + \frac{1}{2} T_t^r \left[(\lambda(r,t) + \nu(r,t))_{,r} + \frac{2(D-2)}{r} \right] \\ = 0, \end{aligned} \quad (2.8a)$$

$$\begin{aligned} T_{r,r}^r + T_{r,t}^t + \left[\frac{1}{2} \nu(r,t)_{,r} + \frac{D-2}{r} \right] T_r^r + \frac{1}{2} [\nu(r,t) + \lambda(r,t)]_{,t} T_r^t \\ - \left[\frac{1}{2} \nu(r,t)_{,r} T_t^t + \frac{D-2}{r} T_{\theta(A)}^{\theta(A)} \right] = 0. \end{aligned} \quad (2.8b)$$

The field equations must now be solved. We adopt here a similar method to that of Sygne²³⁾ which we generalise to accommodate arbitrary dimension.

The system possesses six partial differential equations (which admits two differential identities) and six unknown functions: $\nu(r,t)$, $\lambda(r,t)$ and the four relevant components of the stress-energy tensor. One may therefore either prescribe two of these functions or else they may be determined by other means, such as supplementary matter equations. We assume, for the moment, that T_t^t and T_r^r are the known functions (we will mention more on this issue shortly). They may also be related by an equation of state. We solve for $\lambda(r,t)$ via (2.7a) by noting that (for $D > 2$) it may be written as:

$$-16\pi \frac{r^{D-2}}{D-2} T_t^t = [r^{D-3} w(r,t)]_{,r} \quad (2.9)$$

with $w(r, t) := 1 - e^{-\lambda(r, t)}$. Integrating this equation followed by minor algebraic manipulation gives:

$$e^{-\lambda(r, t)} = 1 + \frac{16\pi}{(D-2)r^{D-3}} \int_{r_0}^r T_t^t(x, t) x^{D-2} dx + \frac{f(t)}{r^{D-3}} =: 1 - \frac{2m(r, t)}{r^{D-3}}. \quad (2.10)$$

Here $f(t)$ is an arbitrary or free function of integration. To avoid a singularity at $r = 0$ one must set this function to zero. However, here we shall keep it for generality.

The metric function, $\nu(r, t)$, is obtained from a linear combination of (2.7a) and (2.7b) as:

$$8\pi [T_t^t - T_r^r] = -\frac{D-2}{2r} e^{-\lambda(r, t)} [\nu(r, t) + \lambda(r, t)]_{,r}, \quad (2.11)$$

which, using (2.10), yields

$$e^{\nu(r, t)} = \left[1 - \frac{2m(r, t)}{r^{D-3}} \right] \exp \left\{ h(t) + \frac{16\pi}{D-2} \int_{r_0}^r \left[\frac{T_r^r(x, t) - T_t^t(x, t)}{x^{D-3} - 2m(x, t)} \right] x^{D-2} dx \right\}. \quad (2.12)$$

The function $h(t)$ is a function of integration which *may* be absorbed in the definition of a new time coordinate via the transformation $\hat{t} = \int \exp [h(t)/2] dt$. This is not always possible as will be discussed in section 4.

The energy flux, T_r^t , may now be defined by the equation (2.7c) and the lateral pressure, T_θ^θ , is defined from the conservation law (2.8a):

$$T_{\theta(A)}^{\theta(A)} := \frac{r}{D-2} \left\{ T_{r,r}^r + T_{r,t}^t + \frac{1}{2} [\nu(r, t) + \lambda(r, t)]_{,t} T_r^t \right. \\ \left. + \left[\frac{1}{2} \nu(r, t)_{,r} + \frac{D-2}{r} \right] T_r^r - \frac{1}{2} \nu(r, t)_{,r} T_t^t \right\}, \quad (2.13)$$

as this must be the lateral pressure if the energy density and parallel pressure are known. At this point, it can be shown that *all* equations and identities are satisfied.

If one is interested in solutions given by specific matter fields, the number of unknowns versus the number of equations may be different than previously specified. For example, if the system to be studied respects absolute spatial isotropy (as in the case of a perfect fluid), then (2.13) becomes a differential equation for $T_r^r = T_\theta^\theta$ which must be solved. The general solution presented above will still contain these as specific cases although other constraints must be met (which may be variationally derived and lead to a determinate system).

Otherwise, one is free to prescribe the two functions. The most physical prescription involves specification of the energy density and one (parallel) pressure. This method is quite useful in examinations of relativistic stellar structure and collapse dynamics where one usually prescribes a reasonable energy and pressure from nuclear theory and studies of plasmas²⁴⁾.

§3. D - dimensional counterparts to known solutions

In this section we construct D-dimensional analogues of some four dimensional solutions. It is useful to study specific solutions since they are illustrative of

how various physical properties in the construction depend on space-time dimension. The first solution we present is previously known. We briefly present it here to illustrate how it may quickly be derived using the method of the previous section.

3.1. Kottler solution

Consider the case of a Kottler (Schwarzschild-(anti) de Sitter) solution²⁵⁾ in D -dimensions. In this situation, the cosmological constant is best viewed as part of T^μ_ν :

$$T^\mu_\nu = -\frac{(D-2)M}{8\pi r^{D-2}}\delta(r)\delta^\mu_t\delta^t_\nu - \frac{1}{8\pi}\Lambda\delta^\mu_\nu, \quad (3.1)$$

M being a constant related to the effective mass of the black hole. Here we are *loosely* using the coordinates r and t to represent the domain within the black hole's event horizon as well as the exterior domain. The integrals in (2.10) and (2.12) may easily be evaluated to yield the D -dimensional counterpart to Kottler's original solution (a coordinate re-scaling is assumed):

$$ds^2 = -\left(1 - \frac{2M}{r^{D-3}} - \frac{2\Lambda r^2}{(D-1)(D-2)}\right) dt^2 + \frac{dr^2}{\left(1 - \frac{2M}{r^{D-3}} - \frac{2\Lambda r^2}{(D-1)(D-2)}\right)} + r^2 d\Omega_{(D-2)}^2. \quad (3.2)$$

The term M/r^{D-3} is a potential or harmonic function in a $D-1$ dimensional Euclidean space which is related to the higher dimensional Newtonian potential.

3.2. Homogeneous, incompressible star

We next study the arbitrary dimensional analogue of the static, constant density star²⁶⁾ as well as derive the corresponding maximum mass/radius relationship. The constant density sphere is of interest since it will yield the upper limit to the surface gravitational red shift for spherical (non black hole) systems in *any* dimension. Interesting examples of $2+1$ and $4+1$ dimensional stellar models may be found in^{27)–29)}.

For the constant density homogeneous sphere we have

$$-T^t_t := \begin{cases} \rho_0 & \text{for } r < a \\ 0 & \text{for } r > a \end{cases} \quad (3.3)$$

with ρ_0 and a some positive constants. All principal pressures are equal and will be denoted by p .

The only non-trivial conservation equation in the static case is (2.8b) from which the condition of hydrostatic equilibrium may be derived:

$$\frac{1}{2}\nu_{,r}(p + \rho) + p_{,r} = 0. \quad (3.4)$$

The pressure is to be derived from this equation. The metric function $\lambda(r)$ is given directly by (2.10)

$$e^{-\lambda(r)} = 1 - \frac{16\pi\rho_0 r^2}{(D-2)(D-1)} =: 1 - qr^2, \quad (3.5)$$

where it can be seen that the mass of the star increases as r^{D-1} .

By using (2.11) and the equation of hydrostatic equilibrium, one obtains:

$$\sigma_{,r} + \lambda(r)_{,r} \frac{\sigma}{2} - \frac{8\pi r}{D-2} e^{\lambda(r)} = 0, \quad (3.6)$$

with $\sigma := (p + \rho_0)^{-1}$.

After much calculation and consideration of boundary conditions at the surface of the star (the pressure must vanish at the stellar boundary), the above equation may be solved for the pressure:

$$p = \rho_0 \left[\frac{(D-3)e^{-\lambda(r)/2} - (D-3)e^{-\lambda(a)/2}}{(D-1)e^{-\lambda(a)/2} - (D-3)e^{-\lambda(r)/2}} \right]. \quad (3.7)$$

As described in the previous section, this is that last piece of information required to construct the solution. The metric function, $\nu(r)$, is now computed directly from (2.12):

$$-g_{tt} := e^{\nu(r)} = \left[\frac{(D-1)e^{-\lambda(a)/2} - (D-3)e^{-\lambda(r)/2}}{(D-1)e^{-\lambda(a)/2} - (D-3)} \right]^2, \quad (3.8)$$

and total fluid mass of the sphere is simply

$$M = \frac{8\pi\rho_0}{(D-2)(D-1)} a^{D-1}. \quad (3.9)$$

At this point it may be noted that the pressure becomes infinite at the same point at which the metric function, g_{tt} , vanishes. This happens when

$$r^2 = \frac{(D-1)^2 M a^2 - 2(D-2)a^{D-1}}{M(D-3)^2}. \quad (3.10)$$

To ensure that this does not occur for any acceptable value of r , we must enforce:

$$\frac{M}{a^{D-3}} < \frac{2(D-2)}{(D-1)^2}. \quad (3.11)$$

This is the upper bound on the gravitational potential of the D -dimensional star and is the generalisation of the $M < \frac{4}{9}a$ law (Buchdahl's theorem³⁰⁾) in four dimensions which applies to any spherical, static stellar model.

Finally, the metric at the boundary takes on the form:

$$ds^2_{|r=a} = -4 \frac{e^{-\lambda(a)}}{[(D-1)e^{-\lambda(a)/2} - (D-3)]^2} dt^2 + e^{\lambda(a)} dr^2 + r^2 d\Omega_{(D-2)}^2, \quad (3.12)$$

from which it can be seen that a coordinate re-scaling:

$$\hat{t} = 2 \left[(D-1)e^{-\lambda(a)/2} - (D-3) \right]^{-1} t, \quad (3.13)$$

allows smooth joining of (3.12) to the $\Lambda \equiv 0$ case of (3.2).

3.3. Anisotropic fluid

Another example is that of the anisotropic fluid star characterized by:

$$\begin{aligned} T_{\mu\nu} &= (\rho + p_{\perp})u_{\mu}u_{\nu} + p_{\parallel}g_{\mu\nu} + (p_{\parallel} - p_{\perp})s_{\mu}s_{\nu}, \\ u^{\alpha}u_{\alpha} &\equiv -1, \quad s^{\alpha}s_{\alpha} \equiv +1, \quad u^{\alpha}s_{\alpha} \equiv 0. \end{aligned} \quad (3.14)$$

In the static case, the metric (2.3) and (3.14) yield

$$\begin{aligned} ds^2 &= -e^{\nu(r)} dt^2 + e^{\lambda(r)} dr^2 + r^2 d\Omega_{(D-2)}^2, \\ u^t &= e^{-\nu(r)/2}, u^r = u^{\theta(A)} \equiv 0, \quad s^r = e^{-\lambda(r)/2}, \quad s^t = s^{\theta(A)} \equiv 0. \end{aligned} \quad (3.15)$$

Now, the static versions of equations (2.10), (2.12) and (2.13) provide the general solution of the problem as:

$$e^{-\lambda(r)} = 1 - \frac{16\pi}{(D-2)r^{D-3}} \int_{r_0}^r \rho(x)x^{D-2} dx + \frac{k_0}{r^{D-3}} =: 1 - \frac{2m(r)}{r^{D-3}}, \quad (3.16a)$$

$$e^{\nu(r)} = \left[1 - \frac{2m(r)}{r^{D-3}} \right] \exp \left\{ c + \frac{16\pi}{D-2} \int_{r_0}^r \left[\frac{p_{\parallel}(x) + \rho(x)}{x^{D-3} - 2m(x)} \right] x^{D-2} dx \right\} \quad (3.16b)$$

$$T_{\theta(A)}^{\theta(A)} := \frac{r}{D-2} \left\{ p_{\parallel,r} + \left[\frac{1}{2} \nu(r)_{,r} + \frac{D-2}{r} \right] p_{\parallel} + \frac{1}{2} \nu(r)_{,r} \rho(r) \right\} =: p_{\perp} \quad (3.16c)$$

Here, k_0 and c are two arbitrary constants of integration. The constant c can be absorbed by the transformation $\hat{t} = e^{c/2}t$ and the constant k can be set to zero to avoid a singularity at the center.

3.4. D -dimensional Neutron star

The neutron star represents a possible end state of a collapsed massive star. The electrons in the matter making up the star are subject to extreme pressures and, via the reaction

$$p^+ + e^- \rightarrow n + \nu, \quad (3.17)$$

are converted to neutrons. The bulk of the energy is carried away by the neutrinos leaving behind a “cold” or degenerate remnant.

The D -dimensional uncertainty principle *per unit volume* yields

$$\prod_{j=1}^{D-1} \Delta k^j = h^{D-1}, \quad (3.18)$$

where k^j is the momentum associated with the k^{th} spatial direction and h is the Planck's constant. By considering a spherical shell of inner radius k and thickness Δk in the phase space, the number of neutrons per unit volume in this shell will be the maximum occupation number times the number of cells with phase volume h^{D-1} :

$$N_n = \frac{4\pi^{\frac{D-1}{2}}}{\Gamma(\frac{D-1}{2})} \frac{k^{D-2} \Delta k}{h^{D-1}}. \quad (3.19)$$

By integrating (3.19) from $k = 0$ to the Fermi momentum, k_F , we get the maximum number of neutrons per unit volume with momentum up to k_F :

$$\mathcal{N} = \frac{4\pi^{\frac{D-1}{2}} k_F^{D-1}}{\Gamma\left(\frac{D-1}{2}\right) (D-1) h^{D-1}}. \quad (3.20)$$

The energy density of the system is simply given by (using (3.19)):

$$\rho = \frac{4\pi^{\frac{D-1}{2}}}{\Gamma\left(\frac{D-1}{2}\right) h^{D-1}} \int_0^{k_F} k^{D-2} (k^2 + m)^{1/2} dk, \quad (3.21)$$

which, in the extremely relativistic limit (acceptable for interior regions of neutron stars¹⁵), yields:

$$\rho = \frac{4\pi^{\frac{D-1}{2}}}{\Gamma\left(\frac{D-1}{2}\right) h^{D-1}} \frac{k_F^D}{D}. \quad (3.22)$$

The pressure of the system is given by differentiating the energy with respect to volume:

$$p = -\frac{dE}{dV} = -\frac{d(\rho/\mathcal{N})}{d(1/\mathcal{N})} = \frac{4\pi^{\frac{D-1}{2}} k^D}{D(D-1)\Gamma\left(\frac{D-1}{2}\right) h^{D-1}} = \frac{\rho}{D-1}. \quad (3.23)$$

This gives the equation of state. Using (3.23) in the conservation law (2.8b) and utilising (2.11) yields the equation of hydrostatic equilibrium for a D -dimensional neutron star (or any highly relativistic, degenerate D -dimensional Fermi gas):

$$\rho_{,r} = \frac{-D(D-3)\rho}{r^{D-2}} \left[1 - \frac{2m(r)}{r^{D-3}}\right]^{-1} \left[\frac{8\pi\rho r^{D-1}}{(D-3)(D-2)(D-1)} + m(r)\right]. \quad (3.24)$$

The densities in the above equation may be eliminated in favour of mass terms, giving an O.D.E. for the mass. By assuming a series solution for $m(r)$, one finds that only one term in the series can contribute to the solution yielding:

$$m(r) = \frac{2(D-1)}{D^2(D-3) + 4(D-1)} r^{D-3}, \quad (3.25a)$$

$$\rho(r) = \frac{(D-3)(D-2)(D-1)}{4\pi[D^2(D-3) + 4(D-1)]} r^{-2}. \quad (3.25b)$$

This second equation indicates that the fall-off properties of the density for the degenerate Fermi gas is *independent of dimension*. The metric functions may now be calculated via (2.10) and (2.12):

$$e^{-\lambda(r)} = 1 - \frac{4(D-1)}{D^2(D-3) + 4(D-1)}, \quad (3.26a)$$

$$e^{\nu(r)} = \left[1 - \frac{4(D-1)}{D^2(D-3) + 4(D-1)}\right] (r/r_0)^{\frac{4}{D}}. \quad (3.26b)$$

Note that for all $D > 3$, the metric given by (3.26a) and (3.26b) is an R -domain type metric.

The above metric derivation is only valid for inner layers of neutron stars where one can make the extreme relativistic approximation. The singularity in the density at the center of the star is also a manifestation due to this approximation. This singularity plagues the known four dimensional model as well where the region near $r = 0$ is usually excised.

§4. The existence of Birkhoff's theorems

One very interesting aspect of spherical symmetry is the existence of Birkhoff's theorem³¹⁾. The original theorem applies to four dimensional systems and states that spherically symmetric vacua are static and are locally equivalent to the Schwarzschild solution. This theorem has since been generalised to include certain matter fields in 4 dimensions including electromagnetic³²⁾,³³⁾ and scalar fields³⁴⁾,³⁵⁾. A study of the differentiability properties required for a well posed staticity theorem may be found in³⁶⁾. Also, some elegant techniques have been employed in the literature regarding higher dimensional versions of the theorem³⁷⁾,³⁸⁾. We briefly present here the conditions required for, as well as a proof of, a D -dimensional Birkhoff's theorem which applies to both vacuum and non-vacuum systems. The theorem presented here generalises previous theorems and below we will extend the theorem to cases where the metric is C_p^0 (piece-wise C^0). The assumptions required for the general theorem to hold are minimal and, most likely, cannot be relaxed, allowing for a most general theorem. We do, however, examine specific cases where conditions may be weakened and the theorem will still hold. We state the theorem as follows:

Theorem: 1 *Let $\mathbf{B} \subset \mathbb{R}^2$ be a convex domain in the $r - t$ plane. Let spherically symmetric metric functions $g_{rr} > 0$ and $g_{tt} < 0$ belong to the class $C^3(\mathbf{B})$ and the stress-energy tensor, T_ν^μ , belong to the class $C^1(\mathbf{B})$. Moreover, let: i) $T_r^t \equiv 0$ and ii) $T_{r,t}^r \equiv 0$. Then, the metric solutions satisfying (2.7a)-(2.7d) must admit an additional Killing vector.*

Proof: By the assumption on the metric functions we can write $g_{tt}(r, t) = -e^{\nu(r,t)}$ and $g_{rr}(r, t) = e^{\lambda(r,t)}$ where $\nu(r, t)$ and $\lambda(r, t)$ are of class C^3 . The identity $T_r^t \equiv 0$ yields, from the equation (2.7c), that

$$\lambda = \lambda(r). \quad (4.1)$$

From this, the equation (2.7a) implies that

$$T_t^t = T_t^t(r). \quad (4.2)$$

Utilising (4.1) in (2.7b) yields:

$$\nu(r, t)_{,r} = \frac{2re^{\lambda(r)}}{D-2} \left\{ 8\pi T_r^r(r) + \frac{(D-2)(D-3)}{2r^2} [1 - e^{-\lambda(r)}] \right\}. \quad (4.3)$$

Therefore, by differentiability with respect to t , we obtain

$$\nu(r, t)_{,r,t} \equiv 0. \quad (4.4)$$

Therefore, in a convex domain \mathbf{B} :

$$\nu(r, t) = \alpha(r) + \beta(t), \quad (4.5)$$

where $\alpha(r)$ and $\beta(t)$ are differentiable functions.

Using (4.1) and (4.5), the metric (2.3) becomes:

$$ds^2 = -e^{\alpha(r)} e^{\beta(t)} dt^2 + e^{\lambda(r)} dr^2 + r^2 d\Omega_{(D-2)}^2. \quad (4.6)$$

By a coordinate transformation,

$$\hat{t} = \int e^{\beta(t)/2} dt, \quad (4.7)$$

the metric in (4.6) reduces to a static one which admits the additional Killing vector $\frac{\partial}{\partial \hat{t}}$. ■

Note that the only physical assumptions involved in the theorem are staticity of the radial pressure and that the system possess no mechanism for radial energy transport. In some circumstances, the above assumptions may be weakened as will be discussed below.

Convexity of the domain is required to address certain situations where the theorem fails though it should seemingly otherwise hold. Consider the following simple counter-example³⁹⁾ depicted in figure 1.

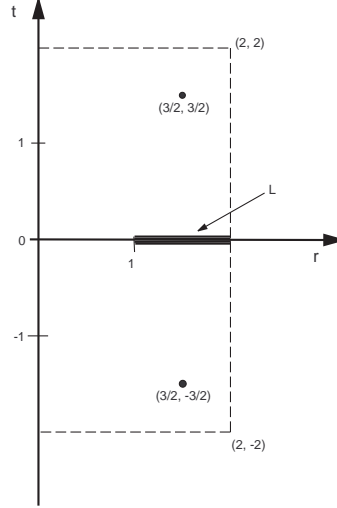


Fig. 1. The non-convex domain provided by considering the area bounded by the dashed line. The line L is removed from the region creating a non-convex domain. In this case, a staticity theorem will not hold.

The line segment, L , is given by $L := \{(r, t) \in \mathbb{R}^2 : 1 \leq r < 2; t \equiv 0\}$. The non-convex domain is furnished by considering the bounded rectangular domain with the line, L , removed: $\mathbf{B} := (\mathbf{0}, \mathbf{2}) \times (-\mathbf{2}, \mathbf{2}) - \mathbf{L}$. Consider now the metric function, $\lambda(r, t)$, given by

$$\lambda(r, t) = \begin{cases} \frac{1}{5}(r-1)^5 & \text{for } 2 > r > 1, t > 0 \\ 0 & \text{otherwise} \end{cases}, \quad (4.8)$$

a C^5 function with derivatives:

$$\lambda(r, t)_{,r} = \begin{cases} (r-1)^4 & \text{for } 2 > r > 1, \quad t > 0 \\ 0 & \text{otherwise} \end{cases} \quad (4.9)$$

and

$$\lambda(r, t)_{,t} \equiv 0. \quad (4.10)$$

Consider the two points $(\frac{3}{2}, \frac{3}{2})$ and $(\frac{3}{2}, -\frac{3}{2})$ in the $r-t$ plane. The function $\lambda(r, t)$ possesses values:

$$\begin{aligned} \lambda\left(\frac{3}{2}, \frac{3}{2}\right) &= \frac{1}{160}, \\ \lambda\left(\frac{3}{2}, -\frac{3}{2}\right) &= 0. \end{aligned}$$

Therefore, although $\lambda(r, t)_{,t} \equiv 0$ in the domain, $\lambda(r, t)$ is *not* independent of time and the metric is non-static. Such arguments may be applied to cases where non-trivial topological features in the manifold will create non-convexity in the $r-t$ domain.

The conditions under which the original, four dimensional vacuum, theorem hold have been weakened to the point of admitting a C^0 metric³⁶⁾ if the metric possesses a separable g_{tt} . We weaken further the conditions here and demonstrate that a D -dimensional metric (not necessarily vacuum) with separable g_{tt} may possess behaviour as pathological as *piece-wise* C^0 (a gravitational shock front) and still be static.

With little loss of generality, the metric under consideration may, in the vicinity of the “jump”, be written as

$$ds^2 = -e^{\alpha(r)} B^2(t) dt^2 + e^{\lambda(r)} dr^2 + r^2 d\Omega_{(D-2)}^2, \quad (4.12)$$

where

$$B(t) := 2 + \text{Sgn}(t). \quad (4.13)$$

The function $\text{Sgn}(t)$ is given by

$$\text{Sgn}(t) := \begin{cases} t/|t| & \text{for } t \neq 0 \\ 0 & \text{for } t = 0. \end{cases} \quad (4.14)$$

One may believe that such a shock front would create singular structure in the manifold since

$$\frac{dB(t)}{dt} = 2\delta(t), \quad (4.15)$$

as well as

$$\begin{aligned} \Gamma_{tt}^t &= \frac{1}{2} \ln(-g_{tt})_{,t} \cdot \\ \Gamma_{tt}^r &= \frac{1}{2} \alpha(r)_{,r} e^{\alpha(r)-\lambda(r)} B^2(t). \end{aligned}$$

Surprisingly, *no* singularity is present in the curvature tensor and such a solution possesses well behaved orthonormal Einstein tensor. This is because all derivatives

of g_{tt} are multiplied by quantities which vanish at $t = 0$ and it has been rigorously proved that $0 \cdot \delta(t) \equiv 0$ ⁴⁰⁾. The transformation in (4.7) is equally well defined as:

$$\begin{aligned}\hat{t} &= \int_{t_0}^t B(\tau) d\tau \quad t_0 < 0 \\ &= \begin{cases} t + |t_0| & \text{for } t_0 \leq 0 \\ 3t + |t_0| & \text{for } t > 0. \end{cases}\end{aligned}\quad (4.16)$$

Notice that the re-scaled time variable, \hat{t} , is continuous at $t = 0$. Such a transformation is therefore admissible and the re-scaled metric is explicitly static. It is unknown if behaviour worse than C_p^0 (such as characteristic fluctuations) may admit a staticity theorem.

It is well known that hyperbolic equations admit exact discontinuous solutions and that Einstein's equations therefore allow for such solutions. Before $t = t_0$ the metric under consideration can be transformed smoothly to a static one. After $t = t_0$ a similar transformation also renders a static metric. At $t = t_0$ there exists a C^0 transformation to a static metric. Therefore, in the entire domain, the metric can be transformed to a static one although the transformation is not the usual C^3 .

The discontinuous metric in the form of (4.12) yields discontinuous orthonormal tetrad as well as the singular Christoffel symbols mentioned above. However, the *Riemann invariants*, which govern the gravitational physics possess a removable discontinuity which may be ignored. The potentially problematic components are:

$$\begin{aligned}R_{\hat{t}\hat{r}\hat{t}\hat{r}} &= \frac{1}{2}e^{-\lambda(r)} \left[\alpha(r)_{,r,r} + \frac{1}{2}\alpha_{,r}^2 - \frac{1}{4}\alpha(r)_{,r}\lambda(r)_{,r} \right], \\ R_{\hat{t}\hat{\theta}\hat{t}\hat{\theta}} &= \frac{1}{2r}\alpha(r)_{,r}e^{-\lambda(r)},\end{aligned}$$

where hatted indices denote quantities calculated in the orthonormal frame. Notice that the Riemann invariants are well behaved and *independent* of the function $B(t)$. These components are all static indicating staticity of the gravitational field and therefore yielding a Birkhoff's theorem.

Generally, a C_0^p metric yields serious singularities in the manifold (see⁴³⁾,⁴⁴⁾ for studies of such systems). However, for the type of metric given by (4.12), singularities are avoided.

Finally, we should mention that results in this section would be unaltered if one relaxes the condition for the space-time to possess a strictly Lorentzian metric. In such case, the metric (4.6) is better written as

$$ds^2 = -e^{\alpha(r)}A(t)dt^2 + e^{\lambda(r)}dr^2 + r^2d\Omega_{(D-2)}^2, \quad (4.17)$$

where the function $A(t)$ may switch sign at one or more values of t , such as $A(t) = t^3$ (see⁴¹⁾⁴²⁾ for examples of such space-times). The Birkhoff's theorem holds both in the Lorentzian branch and the Euclidean branch of the manifold. Also, in the case where the signs of g_{tt} and g_{rr} are both switched, we obtain the generalisation of

Birkhoff's theorem in the T -domain⁴⁵⁾. In this situation, the theorem is a homogeneity theorem as opposed to a staticity theorem since the additional killing vector is space-like.

§5. Concluding remarks

We have considered spherically symmetric solutions to the Einstein field equations with an arbitrary number of spatial dimensions. A reasonably general method has been presented which allows one to solve, at least in quadrature, these equations. The method has been illustrated by quickly and efficiently computing the metric for a D -dimensional black hole with arbitrary cosmological constant, the metric for a D -dimensional constant density star, anisotropic fluid star and neutron star. An upper limit has been placed on the mass/radius ratio of stars of arbitrary dimension. Finally, the minimum general requirements for a D -dimensional Birkhoff's theorem in both vacuum and non-vacuum systems has been presented. To have a rigorous theorem, one must insist on convexity of the domain in question. In certain situations, a staticity theorem may hold even when the metric component, g_{tt} , is only C_p^0 or piece-wise continuous. It may be shown that the theorem also holds for metrics of Euclidean signature as well as inside black hole T -domains.

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